Series

1. Let a be the first term, r be the common ratio and T_r be the r^{th} term .

$$\frac{T_1+T_2+T_3}{T_4+T_5+T_6}=\frac{a+ar+ar^2}{ar^2+ar^3+ar^4}=\frac{1}{r^2}=\frac{4}{9}\Rightarrow r^2=\frac{9}{4}$$

$$T_6=ar^5=15\frac{3}{16}\Rightarrow T_{10}=ar^9=ar^5\left(r^2\right)^2=15\frac{3}{16}\times\left(\frac{9}{4}\right)^2=\frac{19683}{256}=76\frac{227}{256}$$

2. If $x\neq 1$,

$$S=1+\frac{x}{a}(1+x)+\frac{x^2}{a^2}\left(1+x+x^2\right)+\frac{x^3}{a^3}\left(1+x+x^2+x^3\right)+\ldots+\frac{x^{n-1}}{a^{n-1}}\left(1+x+x^2+\ldots+x^{n-1}\right)$$

$$=1+\frac{x}{a}\left(\frac{1-x^2}{1-x}\right)+\frac{x^2}{a^2}\left(\frac{1-x^3}{1-x}\right)+\frac{x^3}{a^3}\left(\frac{1-x^4}{1-x}\right)+\ldots+\frac{x^{n-1}}{a^{n-1}}\left(\frac{1-x^n}{1-x}\right)$$

$$=\frac{1}{1-x}\left[(1-x)+\left(\frac{x}{a}-\frac{x^3}{a}\right)+\left(\frac{x^2}{a^2}-\frac{x^5}{a^2}\right)+\ldots+\left(\frac{x^{n-1}}{a^{n-1}}-\frac{x^{2n-1}}{a^{n-1}}\right)\right]$$

$$=\frac{1}{1-x}\left[\left(1+\frac{x}{a}+\frac{x^2}{a^2}+\ldots+\frac{x^{n-1}}{a^{n-1}}\right)-\left(x+\frac{x^3}{a}+\frac{x^5}{a^2}+\ldots+\frac{x^{2n-1}}{a^{n-1}}\right)\right]=\frac{1}{1-x}\left[\frac{1-\left(\frac{x}{a}\right)^n}{1-\frac{x}{a}}-x\frac{1-\left(\frac{x^2}{a}\right)^n}{1-\frac{x^2}{a}}\right]$$

If $x = 1$ and $a\neq 1$,

$$S=1+\frac{2}{a}+\frac{3}{a^2}+\ldots+\frac{n}{a^{n-1}}=\frac{1}{a^{n-1}}\left[1+2a+3a^2+\ldots+na^{n-1}\right]=\frac{1}{a^{n-1}}\frac{d}{da}\left[a+a^2+a^3+\ldots+a^n\right]$$

$$=\frac{1}{a^{n-1}}\frac{d}{da}\left[\frac{a-a^{n+1}}{1-a}\right]=\frac{1}{a^{n-1}}\left[\frac{1-(n+1)a^n+na^{n+1}}{(1-a)^2}\right]$$

If $x = a$, $a = 1$,

$S =$ ` $+ 2 + \ldots + n = n(n + 1)/2$.

3. $S_i = 1 + r + r^2 + \ldots + r^{i-1} = \dfrac{1-r^i}{1-r}$

$$\therefore\quad \frac{s_1+s_2+\ldots+s_{n-1}}{n}=\frac{1}{n}\sum_{i=1}^{n-1}S_i=\frac{1}{n}\sum_{i=1}^{n-1}\frac{1-r^i}{1-r}=\frac{1}{n(1-r)}\sum_{i=1}^{n-1}\left[1-r^i\right]=\frac{1}{n(1-r)}\left[(n-1)-\frac{r\left(1-r^n\right)}{1-r}\right]$$

$$=\frac{1}{n(1-r)}\left[\frac{n(1-r)-\left(1-r^n\right)}{1-r}\right]=\frac{1}{1-r}\left(1-\frac{s_n}{n}\right)=\frac{n-s_n}{n(1-r)}$$

4.
$$\sum_{r=1}^{n}\frac{2^r-1}{3^{r+1}}=\frac{1}{3}\left[\sum_{r=1}^{n}\left(\frac{2}{3}\right)^r-\sum_{r=1}^{n}\left(\frac{1}{3}\right)^r\right]=\frac{1}{3}\left[\frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}}-\frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}}\right]=1-\left(\frac{2}{3}\right)^{n+1}-\frac{1}{2}\left[1-\left(\frac{1}{3}\right)^{n+1}\right]$$

$$=\frac{1}{2}-\left(\frac{2}{3}\right)^{n+1}+\frac{1}{2}\left(\frac{1}{3}\right)^{n+1}=\frac{1}{2}\left[\frac{3^{n+1}-2^{n+1}-1}{3^{n+1}}\right]$$

5.
$$\left(1+\frac{1}{r}\right)^2+\left(1+\frac{1}{r^3}\right)^2+\left(1+\frac{1}{r^5}\right)^2+\ldots+\left(1+\frac{1}{r^{2n-1}}\right)^2$$

$$= \left[1 + \frac{2}{r} + \left(\frac{1}{r}\right)^2\right] + \left[1 + 2\left(\frac{1}{r}\right)^3 + \left(\frac{1}{r}\right)^6\right] + \ldots + \left[1 + 2\left(\frac{1}{r}\right)^{2n-1} + \left(\frac{1}{r}\right)^{4n-2}\right]$$

$$= n + 2\left[\frac{1}{r} + \left(\frac{1}{r}\right)^3 + \ldots + \left(\frac{1}{r}\right)^{2n-1}\right] + \left[\left(\frac{1}{r}\right)^2 + \left(\frac{1}{r}\right)^6 + \ldots + \left(\frac{1}{r}\right)^{4n-2}\right] = n + 2\left[\frac{1}{r}\frac{1-\left(\frac{1}{r}\right)^{2n}}{1-\left(\frac{1}{r}\right)^2} + \left(\frac{1}{r}\right)^2 \frac{1-\left(\frac{1}{r}\right)^{4n}}{1-\left(\frac{1}{r}\right)^4}\right]$$

$$= n + \frac{2}{r^{2n-1}}\left[\frac{r^{2n}-1}{r^2-1}\right] + \frac{2}{r^{4n-2}}\left[\frac{r^{4n}-1}{r^4-1}\right] \quad \text{, where } \quad r^2 \neq 1.$$

6. Let $f(n) = a + bn + c\,2^n$.

$\therefore$ $f(1) = a + b + 2c = 2$ …. (1), $f(2) = a + 2b + 4c = -1$ …. (2), $f(3) = a + 3b + 8c = -3$ …. (3)

(2) – (1), $b - 2c = -3$ …. (4), (3) – (2), $b + 4c = -3$ …. (5)

(5) – (4), $2c = 1$, $c = 1/2$ …. (6), (6) $\downarrow$ (4), $b = -4$ …. (7)

(6), (7) $\downarrow$ (1), $a - 4 + 1 = 2$, $\therefore$ $a = 5$. $\therefore$ $f(n) = 5 - 4n + (1/2)\,2^n = 5 - 4n + 2^{n-1}$.

Sum to n-term = $\sum_{r=1}^{n}\left(5 - 4r + 2^{r-1}\right) = 5n - 4 \times \frac{n(n+1)}{2} + \frac{2^n - 1}{2-1} = \underline{\underline{2^n - 2n^2 + 3n - 1}}$

7. S(n) = $1 + (1+x)\sin\theta + (1+x+x^2)\sin^2\theta + (1+x+x^2+x^3)\sin^3\theta + \ldots + (1+x+x^2+\ldots+x^{n-1})\sin^{n-1}\theta$

$$= 1 + \left(\frac{1-x^2}{1-x}\right)\sin\theta + \left(\frac{1-x^3}{1-x}\right)\sin^2\theta + \left(\frac{1-x^4}{1-x}\right)\sin^3\theta + \ldots + \left(\frac{1-x^n}{1-x}\right)\sin^{n-1}\theta$$

$$= \frac{1}{1-x}\left[(1-x) + (1-x^2)\sin\theta + (1-x^3)\sin^2\theta + (1-x^4)\sin^3\theta + \ldots + (1-x^n)\sin^{n-1}\theta\right]$$

$$= \frac{1}{1-x}\left[\left(1 + \sin\theta + \sin^2\theta + \sin^3\theta + \ldots + \sin^{n-1}\theta\right) - \left(x + x^2\sin\theta + x^3\sin^2\theta + \ldots + x^n\sin^{n-1}\theta\right)\right]$$

$$= \frac{1}{1-x}\left[\left(\frac{1-\sin^n\theta}{1-\sin\theta}\right) - x\left(\frac{1-(x\sin\theta)^n}{1-x\sin\theta}\right)\right], \quad \text{where} \quad \sin\theta \neq 1 \quad \text{and} \quad x\sin\theta \neq 1.$$

If $|\sin\theta| < 1$ and $|x\sin\theta| < 1$, then $S(\infty) = \frac{1}{1-x}\left[\left(\frac{1}{1-\sin\theta}\right) - x\left(\frac{1}{1-x\sin\theta}\right)\right]$

$$= \frac{1}{1-x}\frac{1 - x\sin\theta - x + x\sin\theta}{(1-\sin\theta)(1-x\sin\theta)} = \frac{1}{1-x}\frac{1-x}{(1-\sin\theta)(1-x\sin\theta)} = \frac{1}{(1-\sin\theta)(1-x\sin\theta)}$$

8. **Method 1**

$$\sqrt[3]{a\sqrt[4]{b\sqrt[3]{a\sqrt[4]{b\ldots}}}} = a^{\frac{1}{3}}b^{\frac{1}{3\times4}}a^{\frac{1}{3\times4\times3}}b^{\frac{1}{3\times4\times3\times4}}\ldots = a^{\frac{1}{3}+\frac{1}{3\times4\times3}+\ldots}b^{\frac{1}{3\times4}+\frac{1}{3\times4\times3\times4}+\ldots} = a\exp\left[\frac{\frac{1}{3}}{1-\frac{1}{12}}\right] \times b\exp\left[\frac{\frac{1}{12}}{1-\frac{1}{12}}\right]$$

$$= a^{\frac{4}{11}}b^{\frac{1}{11}} = \left(ab^4\right)^{\frac{1}{11}}$$

Method 2

Let x be the given expression, then $x = \sqrt[3]{a\sqrt[4]{bx}}$ $\Rightarrow$ x = $= a^{\frac{4}{11}}b^{\frac{1}{11}} = \left(ab^4\right)^{\frac{1}{11}}$.

9. There are r terms in the r^{th} – bracket. The first term in the n^{th} – bracket is the k^{th} – term of the series where $k = [1 + 2 + \ldots + (n-1)] + 1 = \dfrac{n(n-1)}{2} + 1$.

There are n terms in the n^{th} – bracket .

$\therefore$ Value of the first term in the n^{th} – bracket $= A = a + (k-1)d = 1 + \left[\dfrac{n(n-1)}{2} + 1 - 1\right] \times 2 = n^2 - n + 1$

$\therefore$ Sum of the terms in the n^{th} – bracket $= [2A + (n-1)d] \times \dfrac{n}{2} = 2[(n^2 - n + 1) + (n-1) \times 2]\dfrac{n}{2} = n^3$.

Method 1 Sum of the terms in the first n brackets $= \displaystyle\sum_{r=1}^{n} r^3 = \sum_{r=1}^{n} \frac{1}{4}\left[r^2(r+1)^2 - r^2(r-1)^2\right] = \frac{n^2(n+1)^2}{4}$

Method 2 Total no. of terms up to the last term of the n^{th} – bracket $= k' = 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$

$\therefore$ Sum of the first n brackets $= [2a + (k'-1)d]\dfrac{k'}{2} = \left[2 + \left(\dfrac{n(n+1)}{2} - 1\right) \times 2\right]\dfrac{n(n+1)}{2}\dfrac{1}{2} = \dfrac{n^2(n+1)^2}{4}$

10. **(i)** $2S_1 = S_1 + S_1 = (1 + n) + [2 + (n-1)] + [3 + (n-2)] + \ldots + [n + 1] = n(n+1)$. Result follows.

$(x+1)^3 - x^3 = 3x^2 + 3x + 1 \Rightarrow \displaystyle\sum_{x=1}^{n} \left[(x+1)^3 - x^3\right] = 3\sum_{x=1}^{n} x^2 + 3\sum_{x=1}^{n} x + \sum_{x=1}^{n} 1$

$\Rightarrow (n+1)^3 - 1^3 = 3S_2 + 3\left[\dfrac{n(n+1)}{2}\right] + n \Rightarrow S_2 = \dfrac{n(n+1)(2n+1)}{6}$

$(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1 \Rightarrow \displaystyle\sum_{x=1}^{n} \left[(x+1)^4 - x^4\right] = 4\sum_{x=1}^{n} x^3 + 6\sum_{x=1}^{n} x^2 + 4\sum_{x=1}^{n} x + \sum_{x=1}^{n} 1$

$\Rightarrow (n+1)^4 - 1^4 = 4S_3 + 6\left[\dfrac{n(n+1)(2n+1)}{6}\right] + 4\left[\dfrac{n(n+1)}{2}\right] + n \Rightarrow S_3 = \dfrac{n^2(n+1)^2}{4}$

(ii) Consider: $(x+1)^k - x^k = (k+1)x^k + \dfrac{(k+1)k}{1 \times 2}x^{k-1} + \dfrac{(k+1)k(k-1)}{1 \times 2 \times 3}x^{k-2} + \ldots + (k+1)x + 1$

$\displaystyle\sum_{x=1}^{n} \left[(x+1)^{k+1} - x^{k+1}\right] = \sum_{x=1}^{n}(k+1)x^k + \sum_{x=1}^{n}\frac{(k+1)k}{1 \times 2}x^{k-1} + \sum_{x=1}^{n}\frac{(k+1)k(k-1)}{1 \times 2 \times 3}x^{k-2} + \ldots + \sum_{x=1}^{n}(k+1)x + \sum_{x=1}^{n}1$

$\therefore (n+1)^{k+1} - 1 = (k+1)S_k + \dfrac{(k+1)k}{1 \times 2}S_{k-1} + \dfrac{(k+1)k(k-1)}{1 \times 2 \times 3}S_{k-2} + \ldots + (k+1)S_1 + S_0$

(iii) $n\,S_k(n) = n\,(1^k + 2^k + 3^k + \ldots + n^k)$

$= 1^k + [1^k + 2(2^k)] + [1^k + 2^k + 3(3^k)] + \ldots + [1^k + 2^k + 3^k + \ldots + (n-1)^k + n(n^k)]$

$= 1 + [S_k(1) + 2^{k+1}] + [S_k(2) + 3^{k+1}] + \ldots + [S_k(n-1) + n^{k+1}]$

$= S_k(1) + S_k(2) + \ldots + S_k(n-1) + [1^{k+1} + 2^{k+1} + 3^{k+1} + \ldots + n^{k+1}]$

$= S_k(1) + S_k(2) + \ldots + S_k(n-1) + S_{k+1}(n)$

(iv) **(a)** We use induction on k .

From 10 (i), $S_1 = \dfrac{n(n+1)}{2} = \dfrac{1}{2}n^2 + \dfrac{1}{2}n$. $\therefore$ The proposition is true for $k = 1$.

Suppose the proposition holds true for any positive integers less than k and let us prove that the proposition is also true for k .

From (ii) ,

$$S_k = -\frac{k}{2}S_{k-1} - \frac{k}{1\times 2}S_{k-1} - \frac{k(k-1)}{1\times 2\times 3}S_{k-2} - \ldots - S_1 - \frac{S_0}{k+1} + \frac{(n+1)^{k+1}-1}{k+1} \quad \ldots. \quad (*)$$

Since, by inductive hypothesis, S_{k-1} is a polynomial in n of degree k, S_{k-2} s a polynomial in n of degree k – 1 , and so on.

∴ from (*), S_k is indeed a polynomial of degree k + 1 .

∴ By the Principle of mathematical induction, the proposition is true $\forall\, k \in \mathbb{N}$.

(b) $1^k + 2^k + 3^k + \ldots + n^k = An^{k+1} + Bn^k + Cn^{k-1} + \ldots + Ln$ …. (1)

$1^k + 2^k + 3^k + \ldots + (n+1)^k = A(n+1)^{k+1} + B(n+1)^k + C(n+1)^{k-1} + \ldots + L(n+1)$ …. (2)

(2) – (1), $(n+1)^k = A\left[(n+1)^{k+1} - n^k\right] + B\left[(n+1)^{k+1} - n^{k+1}\right] + C\left[(n+1)^k - n^k\right] + \ldots + L$ …. (3)

Compare coefficients of n^k – terms on both sides of (3), ∴ 1 = A [k + 1] $\Rightarrow A = \frac{1}{k+1}$

Compare coefficients of n^{k-1} – terms on both sides of (3),

$$k = A\left[\frac{k(k+1)}{1\times 2}\right] + B[k] = \frac{1}{k+1}\left[\frac{k(k+1)}{1\times 2}\right] + B[k] \quad \Rightarrow B = \frac{1}{2} .$$

(v) S_4, , can be computed using, for instance, the formula in part (ii). It may also proceed as follows :

From part (iv) (b), $S_4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + Cn^3 + Dn^2 + En$.

Putting n = 1, 2, 3 , we get a system of equations in three unknowns C, D, E

$$C + D + E = \frac{3}{10}, \; 8C + 4D + 2E = \frac{13}{5}, \qquad 27 + C + 9D + 3E = \frac{89}{100}$$

Solving, $C = \frac{1}{3}, \quad D = 0, \quad E = -\frac{1}{30} \qquad \therefore\; S_4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

(vi) Using (v) , the validity of the identities are obtained by direct checks.

11. **(i)** $1 + x + x^2 + \ldots + x^n = \frac{x^{n+1}-1}{x-1}$ …. (1)

Differentiate (1) with respect to x ,

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{(x-1)\left[(n+1)x^n\right] - (x^{n+1}-1)(1)}{(x-1)^2} = \frac{(n+1)x^n(x-1) - (x^{n+1}-1)}{(x-1)^2} \quad \ldots. \quad (2)$$

Multiply by x,

$$x + 2x^2 + 3x^3 + \ldots + nx^n = \frac{(n+1)(x^{n+2} - x^{n+1}) - (x^{n+2} - x)}{(x-1)^2} = \frac{nx^{n+2} - (n+1)x^{n+1} + (n+1)x}{(x-1)^2} \quad \ldots. \quad (3)$$

Differentiate (2) with respect to x , $1 + 4x + 9x^2 + \ldots + n^2x^{n-1}$

$$= \frac{(x-1)^2\left\{n(n+2)x^{n+1} - (n+1)^2x^n + (n+1)\right\} - 2(x-1)\left\{nx^{n+2} - (n+1)x^{n+1} + (n+1)x\right\}}{(x-1)^4}$$

$$= \frac{(x-1)\left\{n(n+2)x^{n+1} - (n+1)^2x^n + (n+1)\right\} - 2\left\{nx^{n+2} - (n+1)x^{n+1} + (n+1)x\right\}}{(x-1)^3}$$

$$= \frac{-nx^{n+2} + (n^2+n+1)x^{n+1} - (2n^2+n-1)x^n + n(n+1)x^{n-1} - x - 1}{(x-1)^3} \qquad \ldots. \quad (4)$$

(ii) Let $s = 1^3 + 2^3 x + 3^3 x^2 + \ldots + n^3 x^{n-1}$

Make up the difference:

$$sx - s = n^3 x^n - 3\sum_{k=1}^{n} k^2 x^{k-1} + 3\sum_{k=1}^{n} kx^{k-1} - \sum_{k=1}^{n} x^{k-1}$$

$$s(x-1) = n^3x^n - 3\frac{-nx^{n+2} + (n^2+n+1)x^{n+1} - (2n^2+n-1)x^n + n(n+1)x^{n-1} - x - 1}{(x-1)^3}$$

$$+ 3\frac{(n+1)x^n(x-1) - (x^{n+1}-1)}{(x-1)^2} - \frac{x^n - 1}{x-1}$$

$$\therefore \quad s = \frac{x^n\left[n^3x^3 - (3n^3+3n^2-3n+1)x^2 + (3n^3+6n^2-4)x - (n+1)^3\right] + x^2 + 4x + 1}{(x-1)^4}$$

(iii) $1 + 3x + 5x + \ldots + (2n-1)x^{n-1} = \sum_{k=1}^{n} (2k-1)x^{k-1} = 2\sum_{k=1}^{n} kx^{k-1} - \sum_{k=1}^{n} x^{k-1} = \frac{2nx^n(x-1) - (x+1)(x^n-1)}{(x-1)^2}$

Put $x = \frac{1}{2}$, we have $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \ldots + \frac{2n-1}{2^{n-1}} = \frac{1}{2^{n-1}}\left[3(2^n-1) - 2n\right]$

(iv) In (iii), put $x = -\frac{1}{2}$, $1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \ldots + (-1)^{n-1}\frac{2n-1}{2^{n-1}} = \frac{2^n + (-1)^{n+1}(6n+1)}{9(2^{n-1})}$

12. (i) Let $S = 1 - 2 + 3 - 4 + \ldots + (-1)^{n-1} n$

If $n = 2m$ (even), then $S = [1 - 2] + [2 - 3] + \ldots + [(2m - 1) - 2m] = (-1)m = -m = -\frac{n}{2}$

If $n = 2m + 1$ (odd), then $S = [1 - 2 + 3 - 4 + \ldots - (2m)] + (2m + 1) = -m + (2m + 1)$

$= m + 1 = \frac{n+1}{2}$

(ii) Let $S = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2$

If $n = 2m$ (even), then

$S = (1^2 - 2^2) + (3^2 - 4^2) + \ldots + [(2m-1)^2 - (2m)^2] = -(1 + 2) - (3 + 4) - \ldots - [(2m - 1) + 2m]$

$= -[1 + 2 + \ldots + (2m)] = -\frac{(2m+1)(2m)}{2} = -\frac{n(n+1)}{2}$

If $n = 2m + 1$ (odd), then

$S = [1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2m-1)^2 - (2m)^2] + (2m+1)^2 = -\frac{(2m+1)(2m)}{2} + (2m+1)^2$

$= -\frac{n(n+1)}{2} + n^2 = \frac{n(n+1)}{2}$

(iii) $1^2 - 3^2 + 5^2 - 7^2 + \ldots - (4n-1)^2 = (1^2 - 3^2) + (5^2 - 7^2) + \ldots + \left[(4n-5)^2 - (4n-1)^2\right]$

$= -2(1 + 3) - 2(5 + 7) + \ldots + (-2)[(4n - 5) + (4n - 1)] = -2[1 + 3 + 5 + \ldots + (4n - 1)]$

$$= (-2)\frac{(4n-1+1)n}{2}\times 2 = -8n^2$$

(iv) $2\times1^2 + 3\times2^2 + 4\times3^2 + \ldots + (n+1)n^2 = \sum_{k=1}^{n}(k^3+k^2) = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$

$$= \frac{1}{12}n(n+1)(3n^2+7n+2) = \frac{1}{12}n(n+1)(n+2)(3n+1)$$

13. 1 + 11 + 111 + 1111 + …. (n terms) = $\frac{10-1}{9} + \frac{10^2-1}{9} + \frac{10^3-1}{9} + \ldots + \frac{10^n-1}{9} = \frac{1}{9}\left[10\frac{10^n-1}{9} - n\right]$

14. **(i)** $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \ldots + \frac{1}{n(n+1)} = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{n}-\frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

(ii) $\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \ldots + \frac{1}{n(n+1)(n+2)}$

$$= \frac{1}{2}\left[\left(\frac{1}{1\times2} - \frac{1}{2\times3}\right) + \left(\frac{1}{2\times3} - \frac{1}{3\times4}\right) + \ldots + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)\right] = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right)$$

(iii) Let $u_r = \frac{r}{(2r-1)(2r+1)(2r+3)}$, $v_r = \frac{r(r+1)}{2(2r+1)(2r+3)}, v_{r-1} = \frac{(r-1)r}{2(2r-1)(2r+1)}$

Then $v_r - v_{r-1} = \frac{r(r+1)}{2(2r+1)(2r+3)} - \frac{(r-1)r}{2(2r-1)(2r+1)} = \frac{r(r+1)(2r-1) - (r-1)r(2r+3)}{2(2r-1)(2r+1)(2r+3)}$

$$= \frac{r[(r+1)(2r-1) - (r-1)(2r+3)]}{2(2r-1)(2r+1)(2r+3)} = \frac{r[2r^2+r-1-2r^2-r+3]}{2(2r-1)(2r+1)(2r+3)} = u_r$$

Hence, $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}[v_r - v_{r-1}] = v_n - v_0 = \frac{n(n+1)}{2(2n+1)(2n+3)}$, taking $v_0 = 0$.

15. $s = \sum_{k=1}^{n}\frac{k^4}{4k^2-1} = \frac{1}{16}\sum_{k=1}^{n}\frac{16k^4-1+1}{4k^2-1} = \frac{1}{16}\left[\sum_{k=1}^{n}(4k^2+1) + \frac{1}{2}\sum_{k=1}^{n}\frac{(2k+1)-(2k-1)}{(2k-1)(2k+1)}\right]$

$$= \frac{1}{16}\left[4\times\frac{n(n+1)(2n+1)}{6} + n + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)\right] = \frac{1}{16}\left[\frac{2n(n+1)(2n+1)}{3} + n + \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)\right]$$

$$= \frac{n(n+1)(n^2+n+1)}{6(2n+1)}$$

16. **(i)** $r^2(r^2-1) = r^2(r-1)(r+1) = (r-1)r(r+1)[(r+2)-2] = (r-1)r(r+1)(r+2) - 2(r-1)r(r+1)$

$$s = \sum_{r=1}^{n} r^2(r^2-1) = \sum_{r=1}^{n}(r-1)r(r+1)(r+2) - 2\sum_{r=1}^{n}(r-1)r(r+1) = \sum_{r=1}^{n} u_r - 2\sum_{r=1}^{n} v_r$$

Let $a_r = (r-1)r(r+1)(r+2)(r+3)$, then $u_r = \frac{1}{5}[a_r - a_{r-1}]$

Let $b_r = (r-1)r(r+1)(r+2)$, then $v_r = \frac{1}{4}[b_r - b_{r-1}]$

Hence $s = u_r = \frac{1}{5}\sum_{r=1}^{n}[a_r - a_{r-1}] - 2\times\sum_{r=1}^{n}\frac{1}{4}[b_r - b_{r-1}]$

$$= \frac{1}{5}(n-1)n(n+1)(n+2)(n+3) - \frac{1}{2}(n-1)n(n+1)(n+2) = \frac{1}{10}(n-1)n(n+1)(n+2)(2n+1)$$

(ii) $(r^2+5r+4)(r^2+5r+8) = (r+1)(r+4)[(r+2)(r+3)+2] = (r+1)(r+2)(r+3)(r+4) + 2(r^2+5r+4)$

$= (r + 1)(r + 2)(r + 3)(r + 4) + 2[(r + 2)(r + 3) - 2] = \quad (r + 1)(r + 2)(r + 3)(r + 4) + 2(r + 2)(r + 3) - 4$

$$\therefore \quad \sum_{r=1}^{n} \left(r^2 + 5r + 4\right)\left(r^2 + 5r + 8\right) = \sum_{r=1}^{n} (r+1)(r+2)(r+3)(r+4) + 2\sum_{r=1}^{n} (r+2)(r+3) - 4\sum_{r=1}^{n} 1$$

$$= \frac{1}{5}\sum_{k=1}^{n} \left[(r+1)(r+2)(r+3)(r+4)(r+5) - r(r+1)(r+2)(r+3)(r+4)\right]$$

$$+ \frac{2}{3}\sum_{r=1}^{n} \left[(r+2)(r+3)(r+4) - (r+1)(r+2)(r+3)\right] - 4n$$

$$= \frac{1}{5}\left[(n+1)(n+2)(n+3)(n+4)(n+5) - 1\times 2\times 3\times 4\times 5\right] + \frac{2}{3}\left[(n+2)(n+3)(n+4) - 2\times 3\times 4\right] - 4n$$

$$= \frac{1}{15}(n+2)(n+3)(n+4)\left(3n^2 + 18n + 25\right) - 4n - 40 = \underline{\underline{\frac{1}{15}n\left(3n^4 + 45n^3 + 265n^2 + 765n + 1022\right)}}$$

(iii) By division and then partial fraction, we have:

$$\frac{r^2\left(r^2 - 1\right)}{4r^2 - 1} = \left(\frac{1}{4}r^2 - \frac{3}{16}\right) - \frac{3}{16}\frac{1}{(2r+1)(2r-1)} = \frac{1}{4}r^2 - \frac{3}{16} + \frac{3}{32}\left[\frac{1}{2r+1} - \frac{1}{2r-1}\right]$$

$$\sum_{r=1}^{n}\frac{r^2\left(r^2 - 1\right)}{4r^2 - 1} = \frac{1}{4}\sum_{r=1}^{n} r^2 - \frac{3}{16}\sum_{r=1}^{n} 1 + \frac{3}{32}\sum_{r=1}^{n}\left[\frac{1}{2r+1} - \frac{1}{2r-1}\right] = \frac{1}{4}\times\frac{1}{6}n(n+1)(2n+1) - \frac{3}{16}n + \frac{3}{32}\left[\frac{1}{2n+1} - 1\right]$$

$$= \frac{1}{24}n(n+1)(2n+1) - \frac{3}{16}n + \frac{3n}{16(2n+1)} = \frac{1}{24}n(n+1)(2n+1) - \frac{3}{8}n\frac{n+1}{2n+1} = \underline{\underline{\frac{(n-1)n(n+1)(n+2)}{6(2n+1)}}}$$

(iv) $\dfrac{r^4 + 2r^3 + r^2 - 1}{r^2 + r} = r^2 + r - \dfrac{1}{r^2 + r} = r^2 + r - \left[\dfrac{1}{r+1} - \dfrac{1}{r}\right]$

$$\sum_{r=1}^{n} \frac{r^4 + 2r^3 + r^2 - 1}{r^2 + r} = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r - \sum_{r=1}^{n}\left[\frac{1}{r+1} - \frac{1}{r}\right]$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) + \left[\frac{1}{n+1} - 1\right] = \underline{\underline{\frac{n\left(n^3 + 4n^2 + 2n + 2\right)}{3(n+1)}}}$$

(v) $\dfrac{r^4 + 3r^3 + 2r^2 + 2}{r^2 + 2r} = r + 1 - \left[\dfrac{1}{r+2} - \dfrac{1}{r}\right]$

$$\sum_{r=1}^{n} \frac{r^4 + 3r^3 + 2r^2 + 2}{r^2 + 2r} = \sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 - \sum_{r=1}^{n}\left[\frac{1}{r+2} - \frac{1}{r}\right]$$

$$= \frac{1}{2}n(n+1) + n + \left[\frac{1}{n+2} + \frac{1}{n+1} - \frac{1}{2} - 1\right] = \underline{\underline{\frac{n\left(n^3 + 6n^2 + 14n + 11\right)}{2(n+1)(n+2)}}}$$

(vi) $\dfrac{r^4 + r^2 + 1}{r^4 + r} = 1 + \dfrac{r^2 - r + 1}{r^4 + r} = 1 + \dfrac{r^2 - r + 1}{r(r+1)\left(r^2 - r + 1\right)} = 1 + \dfrac{1}{r(r+1)} = 1 - \left[\dfrac{1}{r+1} - \dfrac{1}{r}\right]$

$$\sum_{r=1}^{n} \frac{r^4 + r^2 + 1}{r^4 + r} = \sum_{r=1}^{n} 1 - \sum_{r=1}^{n}\left[\frac{1}{r+1} - \frac{1}{r}\right] = n - \left[\frac{1}{n+1} - 1\right] = n + 1 - \frac{1}{n+1} = \underline{\underline{\frac{n(n+2)}{n+1}}}$$